



6.4.5 Row-echelon form and rank

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Introduction

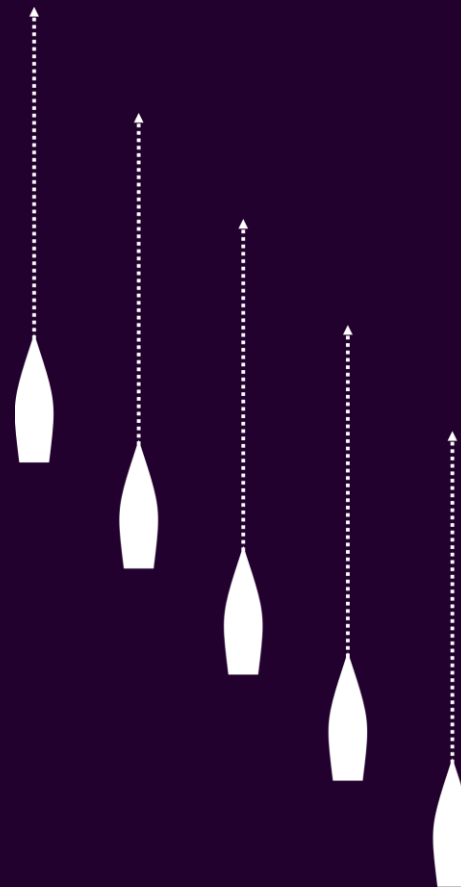
- In this topic, we will
 - Define row-echelon form
 - Define the rank of an augmented matrix
 - Consider some consequences





Echelon

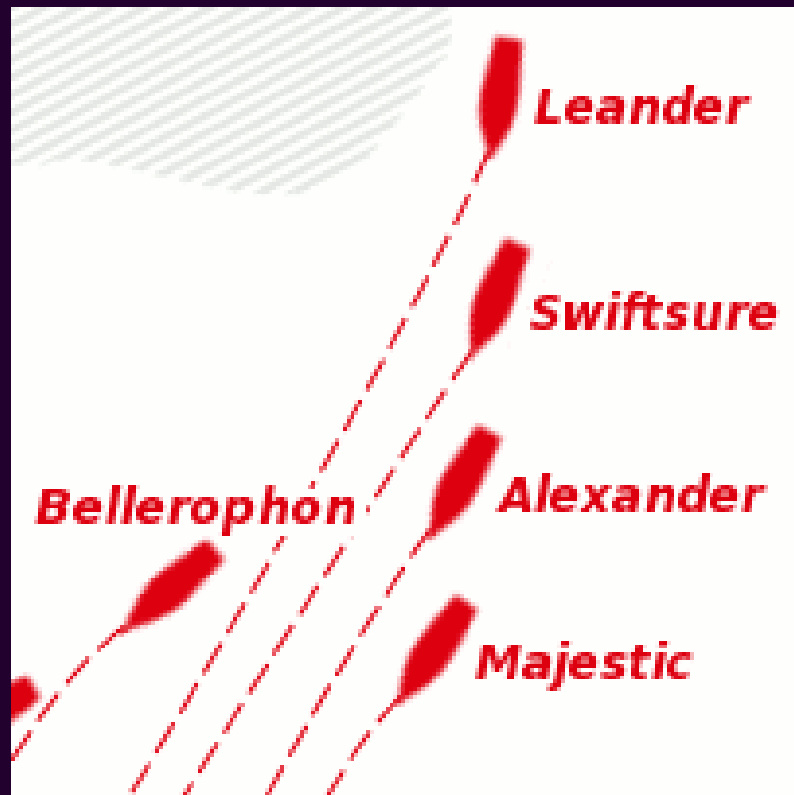
- A fleet of ships is said to be moving *in echelon* if each subsequent ship is behind and to the same side of the preceding one
 - This allowed the fleet to more quickly respond to a threat, in this example, to the right





Echelon

- For example, during the Battle of the Nile, Nelson's ships moved in echelon to better respond to the threat to their right by Napoleon





Augmented matrices

- An augmented matrix is said to be in *row-echelon form* if:
 - All non-zero rows come first,
followed by all zero rows
 - Each successive non-zero row has more leading zeros than the previous non-zero row

$$\left(\begin{array}{ccccc|c} 7.3 & 3.4 & 2.9 & 0 & 5.1 & 9.6 \\ 0 & 0 & 5.5 & 2.5 & 0 & 1.1 \\ 0 & 0 & 0 & -4.9 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{cccc|c} 7.6 & -0.3 & 0 & 0 & 6.8 \\ 0 & 4.7 & 9.2 & 3.8 & -5.4 \\ 0 & 0 & 1.3 & -4.0 & 2.9 \\ 0 & 0 & 0 & 0.3 & 9.8 \\ 0 & 0 & 0 & 0 & 7.6 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$





Augmented matrices

- Consequently, we can make some observations:
 - No two rows can have the same number of leading zeros
 - Any column can contain at most one leading non-zero entry
 - The zero augmented matrix is in row-echelon form
 - An augmented matrix in row-echelon form must be upper triangular

$$\left(\begin{array}{ccccc|c} 7.3 & 3.4 & 2.9 & 0 & 5.1 & 9.6 \\ 0 & 0 & 5.5 & 2.5 & 0 & 1.1 \\ 0 & 0 & 0 & -4.9 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$





Augmented matrices

- Also, if the matrix A has the following properties:
 - A is a square diagonal matrix with no zeros on the diagonal
 - A is a square upper triangular matrix with no zeros on the diagonal

Then $(A \parallel \mathbf{b})$ is an augmented matrix that is already in row-echelon form for any target vector $\mathbf{b}$

$$\left(\begin{array}{cccc|c} -5.9 & 0 & 0 & 0 & 5.1 \\ 0 & 2.7 & 0 & 0 & 0 \\ 0 & 0 & 6.1 & 0 & 3.7 \\ 0 & 0 & 0 & 8.7 & 4.5 \end{array} \right)$$

$$\left(\begin{array}{cccc|c} 1.3 & 0.7 & 0.1 & -0.2 & 0 \\ 0 & 4.5 & 0.3 & 0.1 & 0 \\ 0 & 0 & 3.6 & 0.8 & 0 \\ 0 & 0 & 0 & 2.9 & 0 \end{array} \right)$$





Examples

$$\left(\begin{array}{cc|c} 5 & 8 & -4 \\ 0 & 2 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{cc|c} 5 & 8 & -4 \\ 0 & 2 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{cc|c} 5 & 8 & -4 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{cccc|c} 1.3 & 0.7 & 0.1 & -0.2 & 0 \\ 0 & 4.5 & 0.3 & 0.1 & 0 \\ 0 & 0 & 3.6 & 0.8 & 0 \\ 0 & 0 & 0 & 2.9 & 0 \end{array} \right)$$

$$\left(\begin{array}{cccc|c} 1.3 & 0.7 & 0.1 & -0.2 & 0 \\ 0 & 4.5 & 0.3 & 0.1 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 2.9 & 0 \end{array} \right)$$





Other definitions

- Mathematics courses on linear algebra sometimes require that each non-zero row be divided by the corresponding leading non-zero entry
 - This is inefficient from a computational point-of-view
 - This is not necessary in this course
 - You will not lose any marks if you do,
you will only waste your time
- You may even be made aware of a reduced–row-echelon form
 - This requires even more work on your part
 - Again, you will not lose any marks,
but you will be wasting your time





Rank

- Given a matrix A and a target vector $\mathbf{b}$,
if the augmented matrix $(A \parallel \mathbf{b})$ is in row-echelon form:
 - The *rank* of the augmented matrix is the number of non-zero rows
 - The *rank* of the matrix A is the number of non-zero rows ignoring the right-most column

$$(A \parallel \mathbf{b}) = \left(\begin{array}{ccccc|c} 7.3 & 3.4 & 2.9 & 0 & 5.1 & \blacksquare \\ 0 & 0 & 5.5 & 2.5 & 0 & \blacksquare \\ 0 & 0 & 0 & -4.9 & 1.0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & \blacksquare \end{array} \right)$$

$$\text{rank}(A \parallel \mathbf{b}) = 4$$

$$\text{rank}(A) = 3$$





Rank

- Examples:

$$(B \mid \mathbf{v}) = \left(\begin{array}{cccc|c} 7.6 & -0.3 & 0 & 0 & 6.8 \\ 0 & 4.7 & 9.2 & 3.8 & -5.4 \\ 0 & 0 & 0 & -4.0 & 2.9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} \text{rank}(B \mid \mathbf{v}) = 3 \\ \text{rank}(B) = 3 \end{array}$$

$$(C \mid \mathbf{u}) = \left(\begin{array}{cc|c} 5 & 8 & -4 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} \text{rank}(C \mid \mathbf{u}) = 2 \\ \text{rank}(C) = 1 \end{array}$$





Rank

- If $(A \mid \mathbf{b})$ is in row-echelon form,
the rank of $(A \mid \mathbf{b})$ must either:
 - Equal the rank of A , or
 - Equal to the rank of A plus one

$$\left(\begin{array}{ccccc|c} 7.3 & 3.4 & 2.9 & 0 & 5.1 & 9.6 \\ 0 & 0 & 5.5 & 2.5 & 0 & 1.1 \\ 0 & 0 & 0 & -4.9 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$





Rank

- Some consequences assuming $(A \vdots \mathbf{b})$ is in row-echelon form
 - If A is an $m \times n$ matrix, then $\text{rank}(A) \leq \min\{m, n\}$
 - Also, $\text{rank}(A \vdots \mathbf{b}) \leq \min\{m, n + 1\}$
 - If $\text{rank}(A) = m$, then $\text{rank}(A \vdots \mathbf{b}) = m$





Summary

- Following this topic, you now
 - Understand when an augmented matrix is in row-echelon form
 - Identify augmented matrices that are in row-echelon form
 - Understand the definition of the rank of a matrix
 - You know that $\text{rank}(A \begin{smallmatrix} \vdots \\ \mathbf{b} \end{smallmatrix}) = \text{rank}(A)$ or $\text{rank}(A \begin{smallmatrix} \vdots \\ \mathbf{b} \end{smallmatrix}) = \text{rank}(A) + 1$





References

- [1] https://en.wikipedia.org/wiki/Row_echelon_form
- [2] [https://en.wikipedia.org/wiki/Rank_\(linear_algebra\)](https://en.wikipedia.org/wiki/Rank_(linear_algebra))





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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